

Plane fronted parallel waves in a warm two-component plasma

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A general system of equations is derived, using the 1+3 orthonormal tetrad formalism, describing the influence of a plane-fronted-parallel gravitational wave on a warm relativistic two-component plasma. We focus our attention on phenomena that are induced by terms that are higher order in the gravitational wave amplitude. In particular, it is shown that parametric excitations of ion-acoustic waves takes place, due to these higher order gravitational non-linearities. The implications of the results are discussed.

I. INTRODUCTION

There have been many theoretical investigations on the scattering of electromagnetic waves off gravitational fields, using linearized gravitational wave theory (see e.g. [1]), which is the relevant regime for gravitational wave detectors, or, in general, for distances far away from the gravitational wave source. Much of previous research has been directed at studying effects on vacuum electromagnetic fields, but there has also been some work where the effects of plasmas have been taken into account (see e.g. [2–5] and references therein).

In a recent paper [4] we have taken another approach, starting with an exact gravitational wave solution, but focusing on a weak amplitude (but still non-linear) approximation, and studying the effects induced by gravitational waves in a plasma. As was shown in [4], such interactions give rise to qualitatively new phenomena that are absent in linearized theory. However, the previous work was limited to cold plasmas, and thus did not allow interaction with low-frequency longitudinal waves (although for example Ref. [5] treats the generation of low-frequency plasma modes by *linear* gravitational waves). As will be shown, such waves are more likely to be excited by gravitational waves,¹ due to the frequency matching condition.

We start by generalizing previous equations for a cold two-component plasma to include thermal effects, using the 1+3 orthonormal frame formalism. Within this mathematical framework, the governing plasma equations can be written in a simple 3-dimensional form. They consist of Maxwell's equations with additional charge and current densities characterizing the gravitational effects and a set of fluid equations for a warm plasma. To facilitate the analysis of the non-linear interaction between a plasma and a gravitational wave, we make use of the *plane fronted parallel* (pp) wave solution of Einstein's field equations. We show that for parallel propagation, excitation of ion-acoustic waves can only occur if effects second order in the gravitational wave amplitude are included. The growth rate of the second order instability is determined, and the threshold value for excitation is estimated. Some applications to astrophysics and cosmology are discussed and our results are summarized in the last section of the paper.

II. PRELIMINARIES

The energy-momentum tensor for each particle species is assumed to take the form of a perfect fluid, $T^{ab} = (\mu + p)V^a V^b + pg^{ab}$, where μ is the energy density and p the pressure of each fluid. Splitting the energy-momentum tensor in time- and spacelike parts, using an orthonormal frame $\{e_a, a = 0, \dots, 4\}$, the particle and momentum conservation equations take the form [6]

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¹As compared to the high-frequency excitation process considered in Ref. [4].

$$\mathbf{e}_0(\gamma n) + \nabla \cdot (\gamma n \mathbf{v}) = -\gamma n (\Gamma_{0\alpha}^\alpha + \Gamma_{00}^\alpha v_\alpha + \Gamma_{\beta\alpha}^\alpha v^\beta) , \quad (1a)$$

$$\begin{aligned} (\mu + p) (\mathbf{e}_0 + \mathbf{v} \cdot \nabla) \gamma \mathbf{v} + \gamma^{-1} \nabla p + \gamma \mathbf{v} (\mathbf{e}_0 + \mathbf{v} \cdot \nabla) p \\ = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \gamma(\mu + p) [\Gamma_{00}^\alpha + (\Gamma_{0\beta}^\alpha + \Gamma_{\beta 0}^\alpha) v^\beta + \Gamma_{\beta\gamma}^\alpha v^\beta v^\gamma] \mathbf{e}_\alpha , \end{aligned} \quad (1b)$$

generalizing the results presented in [4] to the case of a warm plasma. Here Γ_{bc}^a are the Ricci rotation coefficients with respect to the ONF $\{\mathbf{e}_a\}$. We have introduced the notation $\mathbf{E} \equiv (E^\alpha) = (E^1, E^2, E^3)$ etc., $\nabla \equiv (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The fluid is assumed to have a four-velocity $(\gamma, \gamma \mathbf{v})$, relative to the orthonormal frame $\{\mathbf{e}_a, a = 0, \dots, 4\}$, where $\gamma \equiv (1 - v^2)^{-1/2}$. Furthermore, $\rho \equiv q\gamma n$ is the charge density.

We follow the approach presented in Refs. [2, 4] for handling gravitational effects in Maxwell's equations. With this, these equations were presented in their generality in [2, 4]. In order to address the issue of how strong gravitational radiation may be involved in generation of EM waves, we look at the pp-waves (for a discussion of this solution, see [7]), in the special case of a linearly polarized plane wave the spacetime metric takes the following form

$$ds^2 = -dt^2 + a(u)^2 dx^2 + b(u)^2 dy^2 + dz^2 , \quad (2)$$

where $u = z - t$, and a and b satisfy $ab_{uu} + a_{uu}b = 0$, and the subscript u denotes a derivative with respect to retarded time. Note that we have chosen a vacuum geometry, i.e. we have omitted the influence of the plasma on the metric.

In order to make physical interpretations simple, we introduce the canonical Lorentz frame

$$\mathbf{e}_0 = \partial_t , \quad \mathbf{e}_1 = a^{-1} \partial_x , \quad \mathbf{e}_2 = b^{-1} \partial_y , \quad \mathbf{e}_3 = \partial_z . \quad (3)$$

With this choice, the gravitational effects in Maxwell's equations are

$$\rho_E = -(\ln ab)_u E^3 , \quad (4a)$$

$$\rho_B = -(\ln ab)_u B^3 , \quad (4b)$$

$$\mathbf{j}_E = -(\ln b)_u (E^1 - B^2) \mathbf{e}_1 - (\ln a)_u (E^2 + B^1) \mathbf{e}_2 - (\ln ab)_u E^3 \mathbf{e}_3 , \quad (4c)$$

$$\mathbf{j}_B = -(\ln b)_u (E^2 + B^1) \mathbf{e}_1 + (\ln a)_u (E^1 - B^2) \mathbf{e}_2 - (\ln ab)_u B^3 \mathbf{e}_3 , \quad (4d)$$

which enters Maxwell's equations as

$$\nabla \cdot \mathbf{E} = \rho_E + \rho_{\text{ch}} , \quad (5a)$$

$$\nabla \cdot \mathbf{B} = \rho_B , \quad (5b)$$

$$\dot{\mathbf{E}} - \nabla \times \mathbf{B} = -\mathbf{j}_E - \mathbf{j} , \quad (5c)$$

$$\dot{\mathbf{B}} + \nabla \times \mathbf{E} = -\mathbf{j}_B , \quad (5d)$$

where $\mathbf{j} \equiv \sum_{\text{p.s.}} q\gamma n \mathbf{v}$ is the current density, and the overdot stands for time derivative.

With respect to the frame choice (3), the conservation equations (1) lead to the following fluid equations:

$$\frac{\partial}{\partial t}(\gamma n) + \nabla \cdot (\gamma n \mathbf{v}) = \gamma n (\ln ab)_u (1 - v_{\parallel}) , \quad (6a)$$

$$\begin{aligned} (\mu + p) \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \gamma \mathbf{v} + \gamma^{-1} \nabla p + \gamma \mathbf{v} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) p \\ = \rho_{\text{ch}}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \gamma(\mu + p) [(\ln a)_u v_1 \mathbf{e}_1 + (\ln b)_u v_2 \mathbf{e}_2] (1 - v_{\parallel}) + \gamma(\mu + p) [(\ln a)_u v_1^2 + (\ln b)_u v_2^2] \mathbf{e}_3 , \end{aligned} \quad (6b)$$

where $v_{\parallel} \equiv v_3$ is the velocity parallel to the gravitational wave propagation direction. These equations should be satisfied for each particle species.

III. LONGITUDINAL WAVE EXCITATIONS

The terms in equations (4)–(6) which generate effects in the direction of propagation of the gravitational wave, i.e. the longitudinal terms, are second order in the gravitational wave amplitude. As was previously shown in [4], these second order terms can give rise to qualitatively new phenomena compared to the linear regime. However, for the process described in [4], the gravitational wave frequency must be equal to the local plasma frequency, which puts rather severe constraints on the possible sources for the gravitational waves. Here we will focus on the excitation of longitudinal waves with much lower frequencies, specifically ion-acoustic waves. For simplicity, the equation of

state is assumed to be an non-relativistic isothermal,² i.e. $p = k_B T n$, k_B being the Boltzmann constant and T the temperature.

As a consequence of the low temperature, the inertial mass term $\mu + p \approx mn$ and it is this approximation that will be used from now on. Furthermore, we consider the regime $\ln(ab) \ll 1$. The background is homogeneous in both fluid species, and thus we can take the zeroth order electromagnetic fields to be zero. Neglecting the effect of charge separation, we obtain the first order solutions to Eqs. (6)

$$n_{\text{gw}}(u) = -n_0 \frac{\ln(ab)}{1 - v_{\text{th}}^2}, \quad v_{\text{gw}}(u) = -v_{\text{th}}^2 \frac{\ln(ab)}{1 - v_{\text{th}}^2}, \quad (7)$$

where n_0 is the background number density and $v_{\text{th}} \equiv \sqrt{k_B T / m}$ the thermal velocity. The above equations hold for both particle species so that the density perturbations are in general different for the two fluids.³

With the above calculation as a prerequisite, we consider the stability of these solutions, i.e. we make the ansätze

$$n(t, z) = n_0 + n_{\text{gw}}(u) + \hat{n}(t, z), \quad v(t, z) = v_{\text{gw}}(u) + \hat{v}(t, z), \quad E(t, z) = \hat{E}(t, z). \quad (8)$$

We linearize in \hat{n} , \hat{v} and \hat{E} , and keep terms up to order $(\ln ab)_u \times (\hat{n}, \hat{v} \text{ or } \hat{E})$. The effect of terms quadratic or higher order in $(\ln ab)_u$ is to modify the equilibrium solution (7) by a factor $\sim 1 + (\ln ab)$. Thus we neglect the corresponding influence on the wave excitation.

Using the above perturbation scheme, we can combine equations (6a) and (6b) to obtain

$$\mathcal{D}\hat{n} = \frac{\partial S}{\partial t} - \frac{n_0 q}{m} \frac{\partial \hat{E}}{\partial z}, \quad (9)$$

where

$$\mathcal{D} \equiv \frac{\partial^2}{\partial t^2} - v_{\text{th}} \frac{\partial^2}{\partial z^2} \quad (10)$$

and

$$S \equiv (n_0 \hat{v} - \hat{n}) \frac{\partial}{\partial t} (\ln ab) - \frac{\partial}{\partial z} (n_{\text{gw}} \hat{v} + \hat{n} v_{\text{gw}}). \quad (11)$$

Note that \mathcal{D} and S in general are different for different particle species.

Acting with $\mathcal{D}_1 \mathcal{D}_2$ on Eq. (5a), we obtain

$$(\mathcal{D}_1 \mathcal{D}_2 + \omega_{p1}^2 \mathcal{D}_2 + \omega_{p2}^2 \mathcal{D}_1) \frac{\partial \hat{E}}{\partial z} = q_1 \mathcal{D}_2 \frac{\partial S_1}{\partial t} + q_2 \mathcal{D}_1 \frac{\partial S_2}{\partial t} + \mathcal{D}_1 \mathcal{D}_2 \left[\hat{E} \frac{\partial}{\partial t} (\ln ab) \right], \quad (12)$$

where the indices 1 and 2 denotes the two fluid species, and $\omega_{p1,2} \equiv [q_{1,2}^2 n_0 / m_{1,2}]^{1/2}$ is the plasma frequency of each particle species. Equation (12) holds for *any* two-fluid plasma with low temperature in both species, and can describe excitation of both high-frequency and low-frequency waves. A process involving high-frequency waves in a cold, one-component plasma, was investigated in [4]. However, in most cases the gravitational wave frequency is lower than the electron plasma frequency. Thus, from now on we will focus on three-wave excitation of low-frequency modes (i.e. all frequencies $\ll \omega_{p,e}$) in an electron-ion plasma. Note that the gravitational wave acts as our pump wave. We are interested in the case of a periodic source of our pump wave, and following [4] we obtain (to second order in an expansion in the amplitude h)

$$(\ln ab) = \frac{1}{2} h^2 \exp[2i\omega_{\text{gw}}(z - t)] + \text{c.c.}, \quad (13)$$

where c.c. denotes the complex conjugate.

²A more general temperature-to-density profile can easily be incorporated into the calculations.

³In principle the electric field due to the possible charge separation should be included in Eq. (7). However, since this effect is proportional to both $(\ln ab)_u$ and v_{th}^2 , it will not influence our main result (17).

The electric field perturbation is assumed to have the form

$$\hat{E} = E_+(t) \exp[i(k_+z - \omega_+t)] + E_-(t) \exp[i(k_-z - \omega_-t)] + \text{c.c.} , \quad (14)$$

and the analogous expressions is assumed to hold for \hat{n} and \hat{v} . The time variations of the amplitudes are slow, as compared to ω_{\pm} . The wave numbers and frequencies satisfy the matching conditions

$$k_+ + k_- = 2\omega_{\text{gw}} , \quad \omega_+ + \omega_- = 2\omega_{\text{gw}} . \quad (15)$$

However, for low-frequency waves, the phase velocities are generally much smaller than unity, which means that equation (15) can be approximated by

$$k_+ = -k_- \equiv k , \quad \omega_+ = \omega_- = \omega_{\text{gw}} . \quad (16)$$

To be consistent, we should simplify the right hand side of Eq. (12) as far as possible, using $v_{\text{th}} \ll 1$, which, for example, implies $S = -\hat{n}\partial(\ln ab)/\partial t$. Inserting the ansätze for \hat{n} , \hat{v} and \hat{E} in Eq. (12), eliminating the density variations using Eq. (9), and applying the approximate matching conditions (16), results in a growth of the amplitudes $\hat{n}, \hat{v}, \hat{E} \propto \exp(\Gamma t)$, with the growth rate

$$\begin{aligned} \Gamma &= 2 \left(\frac{\partial D(\omega_{\text{gw}}, k)}{\partial \omega_{\text{gw}}} \right)^{-1} h^2 \left[\frac{\omega_{\text{gw}}}{k} + \omega_{\text{gw}}^2 \sum_{\text{p.s.}} \frac{\omega_{\text{p}}^2}{(\omega_{\text{gw}}^2 - k^2 v_{\text{th}}^2)^2} \right] \\ &\approx 2 \left(\frac{\partial D(\omega_{\text{gw}}, k)}{\partial \omega_{\text{gw}}} \right)^{-1} h^2 \omega_{\text{gw}}^2 \frac{\omega_{\text{p,i}}^2}{(\omega_{\text{gw}}^2 - k^2 v_{\text{th,i}}^2)^2} \approx h^2 \omega_{\text{gw}} , \end{aligned} \quad (17)$$

where the sum is over particle species and

$$D(\omega, k) = 1 - \sum_{\text{p.s.}} \frac{\omega_{\text{p}}^2}{\omega^2 - k^2 v_{\text{th}}^2} , \quad (18)$$

and the index i refers to the ions.⁴ The scale length $1/k$ of the excited modes, obtained from $D(\omega_{\text{gw}}, k) = 0$, satisfies

$$\frac{v_{\text{th,i}}}{\omega_{\text{gw}}} \leq \frac{1}{k} \leq \frac{c_s}{\omega_{\text{gw}}} , \quad (19)$$

where $c_s \equiv [v_{\text{th,i}}^2 + (m_e/m_i)v_{\text{th,e}}^2]^{1/2}$ is the ion-acoustic velocity.

By including some mechanism of wave damping, a threshold value h_{thr} for excitation may be determined. The most suitable regime for excitation is $T_e > T_i$ in which case ion Landau damping is not effective. Then electron-ion collisions is the main dissipative mechanism, and

$$h_{\text{thr}} \sim \sqrt{\nu_{e-i}/\omega_{\text{gw}}} , \quad (20)$$

which is similar to the results obtained in [4]. Here ν_{e-i} is the electron-ion collision frequency.

Without working out a detailed theory for the saturation mechanism, we note that the wave growth will stop at a level $\hat{v} \leq v_{\text{th,i}}$. This is because the ion-acoustic waves become strongly nonlinear at this level, implying efficient energy transport away from the originally excited modes.

IV. SUMMARY AND DISCUSSION

In the present paper we have generalized previous equations for a cold plasma in the presence of a strong gravitational wave [4], by including thermal effects. As is well known, thermal effects are important for low frequency plasma phenomena. Since, typically, the time-scales for gravitational waves are long compared to the time-scales of a plasma,

⁴The dispersion function $D(\omega, k)$ can be further reduced using $\omega \ll \omega_{\text{p,e}}$, e denoting the electrons. Applied to the expression for Γ , we obtain the last approximate equality in Eq. (17).

this generalization is important from an applicational point of view. The derived equations provide a framework for investigating strong gravitational pulse effects in warm multi-component plasmas. It was shown that the equations indeed admit generation of ion-acoustic modes, which are not present in the linearized theory (cf. Ref. [5]).

Since our effect is of second order in the gravitational wave amplitude, possible astrophysical applications are most likely to be found close to extreme events, such as binary mergers. The expression for the growth rate is almost identical to that of Ref. [4], where two plasmon decay of a pp-wave was considered. However, a fact that makes the present process more relevant for astrophysical applications is that it in principle can occur for arbitrarily low gravitational wave frequency ω_{gw} , i.e. the plasma frequency is allowed to be much larger than ω_{gw} .

Note that Eq. (7) implies that the gravitational wave directly induces charge separation, provided that the thermal velocities of the particle species are different (which is in general true for ion-electron plasma). This charge separation could be important in the early universe, since it could lead to a weak statistically homogeneous and isotropic electric field. Provided the conductivity of the cosmological plasma remains sufficiently low, conditions which exist during inflation and the subsequent period of reheating, this primordial electric field could survive immediate dissipation and could trigger a period of cosmological magnetogenesis through its interaction with linear gravitational waves [8, 9]. In this way large scale cosmological magnetic field could be generated via physical processes inherent to plasmas and the geometrical nature of spacetime, rather than invoking field theoretical arguments such as the breaking of conformal invariance of electromagnetism.

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